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CHECKS, THEIR USE AND ABUSE.*

BY WILLIAM E. BRECKENRIDGE.

In this paper I shall endeavor to answer the following questions:

1. Is it worth while to check mathematical work systematically?

2. What are some dangers to be avoided in introducing the checking system and making it mandatory upon both teachers and pupils?

The scope of the discussion is restricted to secondary schools with some slight reference to industrial and trade schools. The facts presented are gathered from a four years' experience in the practical use of checks in the classrooms of one of the large high schools of New York City.

Notwithstanding the vigorous paragraphs in the books of Professors Smith and Young urging the use of checks, I know only one school where every pupil is held responsible for checking his work and where the teachers are expected as a department to enforce this responsibility on the pupil. Spasmodic attempts here and there have come to my attention where a teacher would talk rather enthusiastically about checking certain particular kinds of work, but the idea of systematically attending to this matter has not yet been adopted.

It is a common complaint from those who employ high school or college men in mathematical work that it is very rare to find a man trained to habits of accuracy.

The ruling of the Regents of New York State on marking inaccuracies on examinations may be interesting. Dr. Wheelock, Chief of the Examination Division, in an address delivered before the New York Section of the Association last year, stated that it was a rule of his department not to give over 70 per cent. for an answer that was incorrect. The College Entrance Board has no rule, but leaves the marking to the judgment of the individual examiner. The Civil Service Board of New York City deducts somewhat less for a mechanical error than the Regents.

* Read at the annual meeting, Lancaster, Pa.

The attitude of secondary teachers, as far as the author's observation has extended, seems to be rather more lenient in marking inaccuracies than the Regents.

Text-books are improving somewhat in the treatment of checks. Several recent books have ventured to extend the checking system to subjects unchecked in previous books. But there is no book that I have seen that persistently and systematically urges the checking of all work, even in those subjects that admit of checks aside from the answer book.

And yet there is great need of more accurate work in our schools. A habit of accuracy is essential to success in applied mathematics.

No surveyor would venture to proceed with his calculations until he had checked them carefully.

An actuary must not only be sure of his work, but must know what methods of interpolation are least liable to error under given conditions.

In banking, an error of addition may cause the loss of a thousand dollars, if made in a certain column.

The industrial chemist must be able to weigh accurately and check his results. Precise observation as well as accurate computation are demanded here.

I fancy few teachers would deduct much, if anything, on account of an error made in copying down an example, yet such an error in industrial work is as fatal to success as an error in theory. Business men complain that boys are not able to copy a set of numbers accurately.

Of course there is no such thing as absolute accuracy. Even after a problem has been checked there always exists the possibility of the corresponding error, but very rarely does this occur and a habit of accuracy will diminish the probability of its occurrence.

I believe that one of the greatest needs of our mathematical teaching at present is a higher standard of accuracy on the part of teachers and pupils which may be attained largely through a general, systematic and persistent use of checks.

A subject like this needs to be worked out in the classroom. All teachers admit the use of checks in a general way, but you will find upon asking whether they teach them, that they have no plan for their systematic use, that their pupils are not

held strictly responsible for checking work, and that, on the whole, the regular work at present takes the whole time, so that the check is considered as something which is perhaps desirable, but under present conditions rather impractical.

The common remark is "It takes too much time." Of course it takes time and patience to develop habits of accuracy, but is it not better to reduce the quantity of work, if necessary, in order to improve the quality of the essentials?

The tendency is to reduce the quantity of work required and improve the quality of what is left. The omission of infinite series from the list of subjects in advanced algebra by the College Entrance Examination Board, on the advice of the American Mathematical Society, and the entire omission of advanced algebra from the entrance requirements to Columbia are examples of this tendency.

My theme, then, is that a pupil should be taught to check all work until the habit is so firmly fixed that the pupil will not be happy nor satisfied with a piece of work till it is checked.

Most of the theories in regard to checks presented in this paper are familiar to you, but it is their effects on the pupils as observed during four years of experiments in the classroom that furnish my material for new arguments for the general adoption of the checking system.

Mathematical work should be checked for two reasons, first, the effect on the pupil and second, the effect on the quality of work done. A boy who has developed a habit of reviewing and examining all work with respect to its accuracy has a self-confidence that is difficult to develop in any other way.

On the other hand, it is very depressing to a student to be in constant doubt as to the accuracy of his work and very injurious to repeat his errors, day after day, without suspecting that the errors exist. He becomes confirmed in bad habits. He may make errors, but he should know that he has made them and not think his work right when it is wrong.

Another reason for checking is that it does secure greater accuracy. Of course checks are not the only means of securing accuracy; skillful teaching and drill are essentials. But it is the general opinion of the teachers who have tried systematic checking that the time taken is well spent.

Another effect on the quality of the work is that incidentally

many important points of theory are taught which are not likely to receive as much emphasis unless the work is checked.

The distinction between the general values of the letters in a polynomial and the unique value of the unknown in a linear equation is usually unknown to a pupil who does not check his work and find out that general values will not satisfy the equation. Extraneous roots are not so well understood as they would be if every root were checked. The convention of taking only the principal root of a radical is not generally understood even by writers of text-books. At least two of the more recent books bring this out clearly. All the others would be compelled to do so, if they should check their results. This is especially important in the problems in quadratic form involving radicals.

If graphs are checked by algebraic solutions, the one-to-one correspondence between algebra and geometry may be illustrated.

The present ambiguous use of zero both as the elementary symbol and as an infinitesimal is illustrated in checking division by avoiding the use of such particular values as will render the division equal to zero.

In trigonometry checks will illustrate the approximate nature of results from logarithmic tables, the calculation of errors, and their proper distribution.

Should all work be checked? Yes; understanding that sometimes work done under the teacher's immediate supervision to emphasize particular points of the subject is sufficiently checked by the teacher's supervision, provided always that the pupil knows how to check it.

The home work should certainly be checked, also all tests and class work not under the immediate supervision of the teacher and often even that to make sure that the student knows how to handle the check.

Continuity and persistency throughout all kinds of work are essential to the development of the proper attitude of the pupil toward checking.

Professor Bain, in his chapter on habit, says: "Continuity of training is the great means of making the nervous system act infallibly right." This applies to checking as well as to drill.

What are the characteristics of a good check?

It should be independent of authority outside of the student himself, such a check as would make the presence of the teacher or answer book unnecessary. Without this quality, the student has no method of testing his results after he leaves school.

Again, the best check tests the problem by some other process than repeating the work already done. The mind tends to act again through the former pathway of discharge and is likely to repeat the error if the same route is taken that was travelled at first.

A chemist in a leading industrial chemical company says that it is a principle of quantitative analysis that chemical work repeated within twenty-four hours is of very little value. In careful experiments, at least a day is allowed to elapse before repeating the work. This same chemist uses an independent method of checking his weights. He counts the value of the holes from which he took his weights and then counts his weights in the scale pan. One checks the other so far as the correct observation and adding of the weights is concerned.

Hence subtraction is better checked by adding subtrahend and remainder or by substituting numerical values than by going over the work of subtraction again.

In trigonometry the solution of a triangle when two sides and the included angle are given is best checked by a sine formula not previously used in the problem. If the third side c has been found by using A , use the sine formula involving B and C for the check.

The ideal check should be readily applied. When the check becomes tedious and difficult of application, much of its value is lost. The substitution of larger numerical values than unity for the letters in checking multiplication is usually unnecessary since nearly all errors are made in the coefficients.

When several checks are possible one should be selected which is most readily applied and which is most effective in detecting errors. Pupils may be allowed to use their judgment and determine by competitive tests in the class which check is most quickly applied and which is most likely to detect errors.

The particular problem should be studied with respect to its special difficulties and the check used which will detect errors that are likely to occur. For example, the substitution of numerical values in the case of factoring involving a common

monomial factor is not so good as multiplying since if the letters are each taken equal to unity, the exponents are not checked and errors are likely to occur in exponents in this case. The multiplication check is more rapid than the substitution of larger values than unity for each letter.

What are the different kinds of checks? First there is the answer book. I know at least one school of high mathematical scholarship where the answer book is put into the hands of pupils in elementary algebra and continued throughout the course.

Another check is the partial answer. For example, a problem in square root has the result given to two figures when three or four are required.

Other forms of checks are the substitution of numerical values, symmetry, homogeneity, approximate estimates, repeating the work, and, for arithmetical computations, the slide rule or a more complicated counting machine.

What are the best checks for the several kinds of work in algebra and trigonometry?

For the four fundamental operations with polynomials, the substitution of particular values is the best check even in subtraction where it is possible to add the remainder and subtract, and in division where the quotient may be multiplied by the divisor. The reason is, first, this check is more quickly applied. A trial by the watch will convince of this if the letters are each given the value of unity. This is usually sufficient since errors are not likely to occur in exponents.

Second, this check is more likely to be done by the student than the other, especially on home work. It is something which can be written down by the pupils and quickly inspected by the teacher.

The other check should be explained, but this one should be used in nearly all cases.

Some examples in division should be given which have a remainder in order to give the student a full understanding of the relation between dividend, divisor, quotient and remainder. The ordinary text-book in algebra gives only examples where the division is exact. From constant use of such examples the pupils come to think that the test of accurate work is that there is no remainder—as they say, “It comes out even.”

In factoring, the monominal factor case should be checked by multiplication, since here errors are likely to occur in the exponents. Other cases, however, lend themselves very readily to the numerical substitution check.

Of course care must be taken to avoid those particular values which make a factor zero.

In the removal of complicated systems of parentheses two methods of work may be taught, one as a check on the other. Whether the method of removal beginning at the outermost parentheses is favored by the teacher, one may be used as a check on the other, or, better yet, a third method of removing all the signs at once may be taught and used as check on either of the other two methods. As far as I know, this last method, while not new, is not given in any text-book; so I venture to illustrate it as follows:

$$d + \{ 2a - \overset{1}{[} 3a - \overset{2}{(} 4a - \overset{3}{5}a - 6 \overset{3}{)}] \}$$

Neglect the positive signs of aggregation. Number the other signs according to the number of minus signs which affect the terms immediately within the sign of aggregation. Remove all signs of aggregation in one step, changing the signs of the terms immediately within the odd-numbered signs of aggregation and leaving unchanged the signs of those immediately within those with even numbers.

$$\begin{aligned} a + 2a - 3a + 4a - 5a + 6 \\ = -a + 6 \end{aligned}$$

A more complicated example involving two distinct systems of parentheses follows:

$$\begin{aligned} a + \{ 3a - \overset{1}{[} 2a - \overset{2}{(} 6a - 1 \overset{2}{)} + 4a - \\ \qquad \qquad \qquad \overset{2}{(} 7a + 1 \overset{1}{)}] \} - \overset{1}{[} 3a - \overset{2}{(} 4a - 1 \overset{2}{)}] \\ = a + 3a - 2a + 6a - 1 - 4a + 7a + 1 - 3a + 4a - 1 \\ = 12a - 1 \end{aligned}$$

A similar method holds when the parentheses have coefficients.

This system is as readily mastered as either of the others. The truth about this kind of work is, however, that while it is

still required by examining boards, in my opinion it has not enough value, to justify it as a topic in the syllabus. The syllabus in algebra of the Regents of New York State reads: "Examples free from ingenious repetition of complications should be selected for practice."

A study of recent papers set by the Regents or by the College Entrance Board shows a tendency to omit tests on complicated work of this kind.

In numerical linear equations, of course, substitution of the value of the unknown is the only check. The distinction between this unique value of X and the general values of A , B , C , etc., in a polynomial should be emphasized at this point by means of the check. Some students in the class will check by substituting unity for X . Let them find their error and teach them the reason for the distinction.

Checks for literal equations may be either the substitution of numerical values for the known letters or the substitution of the root itself. When the substitution of the root is tedious, the particular numerical value should be used. The check for problems involving the making of equations is the analysis of the original problem with the results secured. Substitution in the equation of course merely checks the solution of the equation and not its correct formation.

In H.C.F. and L.C.M., care should be taken to avoid the substitution of numerical values, since such substitution does not check the work. The H.C.F. of $a + b$ and $a - b$ is 1, but if $a = 4$ and $b = 2$, the H.C.F. is 2. In this work the only check good for general use is to repeat the work. However, if only two numbers are involved and both H.C.F. and L.C.M. are required, a useful test of accuracy is the formula $\text{H.C.F.} \times \text{L.C.M.} = \text{the product of the two numbers}$.

All through the reduction of fractions, additions, subtraction, multiplication, division and complex fractions, the check of numerical values is effective. Here particularly text-books fail in sufficiently emphasizing the importance of testing the difficult work of handling fractions. More students fail right here in their algebra than in any other part of the subject. Careful drill and checking of work are very necessary.

When graphs are introduced along with simultaneous equations, the geometric solution should be checked by the algebraic

result. The one-to-one correspondence of geometry and algebra may be illustrated by the use of the check at this point.

The numerical check applies also to involution and evolution. In evolution again, as in division, examples should be given that do not "come out even," as the students say.

For the theory of exponents, the answer book or repeating the work are the proper checks. The answer book is useful also in the transformation of radicals. Radical equations need to be checked carefully to detect extraneous roots. This subject of extraneous roots and that of equivalent equations are too little treated in most text-books.

If the authors of most of our text-books in elementary algebra should check the equations in quadratic form involving the radicals, with the answers from their own answer books, and should use the convention regarding principal roots, a great many of their results would be found to be wrong, since they would not check.

In the substitution case of simultaneous quadratics it is a sufficient test of the work to substitute in one equation only, taking care not to use the one from which the second unknown was secured. Any error in the problem that is likely to occur will appear by this process.

In the homogeneous case, it is sufficient to check the first and third sets of answers, since the checks for the other sets will be a repetition of this work. This work is difficult and somewhat tedious at times, but it is well worth while both for the sake of accuracy in the problem itself and for the review of radicals and imaginaries that it affords.

For the progressions where the work is not too tedious, the series should be written out. The value of the repeating decimal may, of course, be checked by the converse process of reducing the fraction to a decimal.

Partial fractions should be checked by addition. Complex numbers may be illustrated by graphs and checked algebraically.

Determinants applied to equations may be tested by the algebraic solution.

For Horner's Method there are the checks of substitution obtaining approximately zero for the function, or the theory of the relation of the roots and coefficients. This theory applies also to quadratics.

Incommensurable roots obtained by graphs may be tested by substitution.

In trigonometry all text-books check the solution of the triangle when three sides are given, but very few indicate what methods should be used in other cases.

My experience with trigonometry is that it is well worth while to check all triangles right and oblique. The check for a right triangle is the formula $b^2 = c^2 - a^2$ (c being the hypotenuse) in the fractured form $b^2 = (c + a)(c - a)$.

For the oblique triangle, given two sides and the included angle, the sine formula should be used which involves the third side of the triangle and is a formula not previously used in the work; *e. g.*, if we have given the angle C and sides a and b , we may find the side c by using the sine formula involving the letters c and a . If so, we must check by the sine formula involving the letters c and b . When three sides are given, of course the check is the sum of the angles $= 180^\circ$.

In the other cases of oblique triangles the tangent formula is the only reliable check formula. This is somewhat tedious in application, but even so, it is worth while to spend the necessary time upon it, since it furnishes a more reliable test of accuracy than going over the work a second time.

The difficulty with the check by repeating the operation is that the mind tends to act again in the same way it has once acted. Hence an error once made may not be readily found. In the case of the solution of the triangle when three sides are given, the amount of error should be clearly stated on the student's paper and the distribution among three angles clearly indicated. Right here a good transit is a great help in clarifying the subject of possible approximation to accuracy. When a boy has used a transit to some extent he gains an increased respect for accuracy both in seeing and in computing. He learns that a result that is inaccurate is worthless whether it comes from careless reading of the instrument or from careless computation.

Trigonometric equations should be checked by substitution in the original equation. Here again when radicals enter, extraneous roots creep in. Hence the check is a necessary part of the work.

The accuracy of the regular formulas may be tested by the substitution of a particular value for the angle such as 30° , 45°

or 60° whose functions are well known. Functions of the double angle may be tested by assuming $x = 30^\circ$, while for the half angle x should be taken $= 60^\circ$.

So much for checks of algebra and trigonometry in secondary schools.

Just a word in regard to the use of the check in continuation schools.

The kind of men who work in our evening schools, especially those who are in the industrial or the trade schools, are particularly appreciative of methods that make them more efficient workmen. Nearly all of them are employed during the day and feel keenly any advantage of knowledge which makes them better workmen or which enables them to secure employment at an advanced salary. I have been very intimately associated with this class of young men for the past three years. With them I have used all the checks which I have described in algebra and trigonometry and in addition work with the slide rule. If checking were too tedious and not worth the time spent these men would very quickly have shown it, but the fact is they have been very proud of their newly acquired ability to test their own work. These men judge a thing entirely from the standpoint of its use to them. If you have an experiment in teaching and wish to judge it from the standpoint of interest, no better place can be found in which to test it than the classes of men in evening high and trade schools.

But there are certain things to be guarded against in the use of checks. Their abuse may lead to disaster in the scholarship of a class.

First, avoid making the check tedious. Rather review the direct operation or use the answer book than cumber the subject too much with needlessly long operations which take much longer than the problem itself. Just when a check becomes tedious is a fine point for the teacher's judgment to decide.

Second, the check is only one element in obtaining accuracy. Teaching and drill must not be neglected. A student must know how to obtain his results before he will have any results to check. I have observed teachers using so much time during the recitation for checks that there was not enough taken for teaching and drill on the processes. This, however, is no argument against the use of checks, but rather a poor adjustment of the plan for teaching the lesson.

A third difficulty in using checks is the spasmodic nature of some of the experiments. A teacher becomes partially inspired with the idea of checking work and without thinking out the whole matter carefully enough, he proceeds to check work now and then. He teaches his class how to check, but does not immediately hold them responsible for doing so. At first a class must be held up to checking in the same way they would be held to any other kind of work. It involves extra labor and time spent on the home work, which few boys will do at first unless required by the teacher. After the student is more mature, and has seen the value of checking, he will demand methods of testing his work and will not need to be held up to it.

This latter stage, however, does not come to any great extent until a boy is actually employed in real work outside the school.

Again some pupils will hand in daily work that looks as though it were checked, but on test work the results are wrong. A search for the reason for this reveals the fact that the student has "made a bluff at the check," as he says. He has not really tested his results on his daily work, but put down something that looks like a check to deceive the teacher. For this reason it is necessary to observe carefully the result of the test work and if wrong, to mark such dishonest work very severely. When a boy sees that he cannot deceive by his dishonesty, he will usually abandon his attempt.

For the prevention of these abuses I have found certain precautions effective.

First, a certain per cent. should be deducted for failure to check work. This should be decided upon by the department and as rigidly used as possible. In my own school this is 20 per cent. That is, we deduct 20 per cent. from all work that does not show the check.

In a large department there will be some classes at the introduction of such a standard that will not be able to attain it at once. This will be true because of the poor teaching in that particular class. In such cases some elasticity should be allowed in marking, but such elasticity should be noted by the head of the department and the teacher brought up to the standard as soon as possible.

I do not contend for a fixed 20 per cent. Perhaps this is too little, perhaps too much. One teacher in my school holds his

classes up to 25 per cent. But I do contend that some fixed per cent. should be adopted and the teacher as well as the pupils held up for it.

Another device for securing the use of checks is to have a uniform method of ruling paper, so as to reserve a column for checks. If a good-sized column at the right of the paper is ruled and reserved for checks only, the student will be apt to fill out the blank space with the checks.

Too long lessons must be avoided. If checking is required, fewer problems must be assigned both on home work and for tests. It is better to have a few well-selected examples done and checked than to have more done without the student's knowing whether they are right or wrong.

The value of a thing in the long run is determined by common sense. Common sense is a hard thing to define. Perhaps as good a definition as any is one given at a meeting of our New York Section last year by a member of the Board of Examiners of New York City. He said: "Common sense is that which appeals to the non-professional."

Judged by this standard checks seem to be very favorably received.

In conclusion I wish to emphasize the importance of teachers of mathematics insisting upon checks in all mathematical calculations for the sake of (1) the good habits and the feeling of self-confidence in the pupil, (2) knowledge of certain points of theory not gained so well in other ways, (3) increased accuracy in work.

The great thing to be aimed at is to develop in the student such habits of invariably reviewing his work and considering his result in the light of the check that best suits it, that he will not be happy nor content until he has done this whenever a new problem presents itself. If this is done, systematically and invariably, there will be developed a degree of accuracy and a feeling of self-confidence that cannot be obtained in any other way. It is a result which is carried over into other affairs of life.

I hope to see an organized effort on the part of our association to request those who set the examinations for the C.E.E. Board, the Regents of New York State, and the colleges to require checks of all problems on the examinations which they

set and to deduct a fixed percentage for failure to do this. They should insist that the candidate should show on his paper that he understands how to check his work. Of course this will not be done without adequate notice to the schools.

When it is done we shall have text-books adequate for such a requirement, teachers will systematically teach checks, and the result will be a greatly increased accuracy and efficiency in our mathematical work.

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THE AIMS IN TEACHING GEOMETRY AND HOW TO ATTAIN THEM.*

By W. E. BOND.

The aims in teaching geometry should be, according to my views:

1. That the pupil should acquire an accurate, thorough knowledge of geometrical truths.
2. That he should develop the power of original, logical, geometrical reasoning.
3. That he should acquire a habit of thought which will give him a practical sagacity; which will develop his judgment, increase his resourcefulness, and fit him to cope more successfully with the many and varied problems of his after life; which will teach him to take a many-sided view of things, so that if the avenue of attack is blocked, he shall be able to promptly, cheerfully and successfully attack from another quarter.

In my opinion the last-mentioned aim is by far the most important; the first, as an end in itself, I deem the least important of the three.

Having stated the aims to be attained, I now turn to a discussion of the difficulties which, in my own experience, have presented themselves, and have tended to hinder the attainment of these aims.

In my early experience as a teacher of geometry the first difficulty I met was the seeming lost or dazed condition of so

* Read at the meeting of the Syracuse Section, December 29, 1908.